Schwartz 2.6 (a)

$$\int_{-\infty}^{\infty} dk^0 \ \delta(k^2-m^2)\theta(k^0) = \int_0^{\infty} dk^0 \ \delta(k^2-m^2)$$

$$\delta(k^2-m^2) = \delta(k^{0^2} - |\vec{k}|^2 - m^2)$$

$$= \left| \frac{d[k^{0^2} - |\vec{k}|^2 - m^2]}{dk^0} \right|^{-1}_{k^*} \delta(k^0 - k^*)$$

$$k^* = \sqrt{|\vec{k}|^2 + m^2}$$

$$= \frac{1}{2k^*} \delta(k^0 - k^*)$$

$$\Rightarrow \int_{-\infty}^{\infty} dk^0 \ \delta(k^2-m^2)\theta(k^0) = \int_0^{\infty} dk^0 \ \delta(k^2-m^2)$$

$$= \int_0^{\infty} dk^0 \frac{1}{2k^*} \delta(k^0 - k^*)$$

$$= \frac{1}{2k^*} = \boxed{\frac{1}{2\omega_k}}$$

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(b)

$$|d^4k'| = \det[J]\,|d^4k|$$

$$= \det[\Lambda^\mu{}_\nu]\,|d^4k|$$

where $k'^\mu = \Lambda^\mu{}_\nu k^\nu$.

$$\Lambda^T \eta \Lambda = \eta \Rightarrow |\Lambda^T \eta \Lambda| = \eta$$

$$\Rightarrow |\Lambda^T|\,|\Lambda| = 1$$

If Λ is a rotation, then obviously $|\Lambda| = 1$

If Λ is a boost, then $\Lambda^T = \Lambda$, $|\Lambda||\Lambda| = 1 \Rightarrow |\Lambda| = 1$.

If Λ is parity or time reversal, then

$$\Lambda = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad \text{or} \quad \Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

and $|\Lambda| = 1$.

Since general Λ can be written as a combination of rotation, boost, parity and time reversal, $|\Lambda| = 1$, and

$$|d^4k'| = |\Lambda|\,|d^4k| = |d^4k|.$$

(c) k^μ is a vector, so it's additive, we know

under Lorentz transformation, $|\vec{k}| \to \gamma|\vec{k}|$, and

$k^0 \to \gamma k^0$, so $|d^3\vec{k}| \to \gamma|d^3\vec{k}|$ and $k^0 \to \gamma k^0$.

Thus $$\left|\frac{d^3\vec{k}}{k^0}\right| \to \frac{\gamma|d^3\vec{k}|}{\gamma k^0} = \left|\frac{d^3\vec{k}}{k^0}\right|$$

Thus $\frac{d^3\vec{k}}{w_k}$ is Lorentz invariant.

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